Jackson

3.2. (a) $\sigma = \dfrac{Q}{4\pi R^2}$,

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial \hat{n}} = -\varepsilon_0 \frac{\partial \Phi}{\partial(-r)} = \varepsilon_0 \frac{\partial \Phi}{\partial r}.$$

$$\sigma(R) = \varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_R = \frac{Q}{4\pi R^2}.$$

$$\Rightarrow \left.\frac{\partial \Phi}{\partial r}\right|_R = \frac{Q}{4\pi R^2 \varepsilon_0}.$$

In region containing 0, minus powers of r must have ~~van~~ vanishing coeffs.

$$\Rightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

$$\frac{\partial \Phi}{\partial r}(r,\theta) = \sum_{l=0}^{\infty} A_l \, l \, r^{l-1} P_l(\cos\theta).$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_R(\theta) = \sum_{l=0}^{\infty} A_l \, l \, R^{l-1} P_l(\cos\theta)$$

$$\int_0^{\pi} \left[\frac{\partial \Phi}{\partial r}\right]\Big|_R P_l[\cos\theta] \sin\theta \, d\theta = l A_l R^{l-1} \frac{2}{2l+1}$$

$$= \frac{Q}{4\pi\varepsilon_0 R^2} \int_{\alpha}^{\pi} P_l[\cos\theta] \sin\theta \, d\theta$$

Let $x = \cos\theta$, $dx = -\sin\theta \, d\theta$, we have.

$$l A_l R^{l-1} \frac{2}{2l+1} = \frac{Q}{4\pi \varepsilon_0 R^2} \int_{-1}^{\cos\alpha} P_l[x]\, dx$$

By Rodriguez's formula, $\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} = (2l+1) P_l$.

(Jackson 3.28).

Thus

$$l A_l R^{l-1} \frac{2}{2l+1} = \frac{Q}{4\pi \varepsilon_0 R^2} \left[\frac{1}{2l+1}\right] \int_{-1}^{\cos\alpha} \left[\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx}\right] dx$$

$$2 l A_l R^{l-1} = \frac{Q}{4\pi \varepsilon_0 R^2} \Big[P_{l+1} - P_{l-1}\Big]_{-1}^{\cos\alpha}.$$

$$A_l = \frac{Q}{8\pi \varepsilon_0 R^2} \frac{1}{l} \frac{1}{R^{l-1}} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right].$$

$$\Rightarrow \boxed{\Phi = \frac{Q}{8\pi \varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{l} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \frac{r^l}{R^{l+1}} P_l(\cos\theta)}$$

off by some coefficients ?

Davidson Cheng

12.28.2023.

Potential outside $\Phi = \sum \left[A_l r^l + B_l r^{-(l+1)}\right] P_l[\cos\theta]$.

$$\sigma(R) = -\varepsilon_0 \frac{\partial \Phi}{\partial r} = -\varepsilon_0 \frac{\partial \Phi}{\partial R}$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta]$$

$$-\varepsilon_0 \frac{\partial \Phi}{\partial R} = -\varepsilon_0 \sum B_l \left[-(l+1)\right] r^{-(l+2)} P_l[\cos\theta]$$

$$= \varepsilon_0 \sum B_l (l+1) r^{-(l+2)} P_l[\cos\theta] = \frac{Q}{4\pi R^2}$$

$$\sum B_l(l+1) R^{-(l+2)} P_l[\cos\theta] = \frac{Q}{4\pi R^2 \varepsilon_0}$$

$$\cancel{B_l \cos} \, B_l (l+1) R^{-(l+2)} \frac{2}{2l+1} = \frac{Q}{4\pi R^2 \varepsilon_0} \int_{\alpha}^{\pi} P_l[\cos\theta] \sin\theta \, d\theta$$

$$B_l [l+1] R^{-(l+2)} \frac{2}{2l+1} = \frac{Q}{4\pi R^2 \varepsilon_0} \int_{-1}^{\cos\alpha} P_l[x]\, dx.$$

$$= \frac{Q}{4\pi R^2 \varepsilon_0} \frac{1}{2l+1} \int_{-1}^{\cos\alpha} \left[\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx}\right] dx$$

$$B_l [l+1] R^{-(l+2)} = \frac{Q}{8\pi R^2 \varepsilon_0} \left[P_{l+1}[\cos\alpha] - P_{l-1}[\cos\alpha]\right]$$

$$\boxed{B_l = \frac{Q}{8\pi \varepsilon_0} R^l \frac{1}{(l+1)} \left[P_{l+1}[\cos\alpha] - P_{l-1}[\cos\alpha]\right]}$$